REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*	**2. REPORT TYPE**	**3. DATES COVERED** *(From - To)*
31-07-2011	Final Technical	1 Mar 2009 - 30 Apr 2011

4. TITLE AND SUBTITLE
Convection Regularization of High Wavenumbers in Turbulence ANS Shocks

5a. CONTRACT NUMBER

5b. GRANT NUMBER
FA9550-09-1-0158

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
Kamran Mohseni

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Department of Aerospace Engineering Sciences, University of Colorado at Boulder

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Air Force Office of Scientific Research
875 N. Randolph St, Suite 325
Arlington, VA 22203

10. SPONSOR/MONITOR'S ACRONYM(S)
AFOSR

11. SPONSOR/MONITOR'S REPORT NUMBER(S)
AFRL-OSR-VA-TR-2012-0509

12. DISTRIBUTION / AVAILABILITY STATEMENT
Distribution A

13. SUPPLEMENTARY NOTES

14. ABSTRACT
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15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON** Fariba Fahroo
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**	UU		**19b. TELEPHONE NUMBER** *(include area code)*
U	U	U			7036968429

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

CONVECTIVE REGULARIZATION OF
HIGH WAVENUMBERS IN TURBULENCE ANS SHOCKS

AWARD NO: FA9550-09-1-0158

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1 Introduction

This manuscript discusses the recent results in the research project on inviscid regularization techniques with emphasis on applications to compressible gas flows. It is well-established that general systems of quasilinear, first-order hyperbolic PDEs exhibit finite-time blowup of smooth solutions, thus restricting well-posedness of classical solutions in short-time. For instance, with conservation laws, one must generally seek global-in-time solutions in a larger class of discontinuous solutions. Further physically motivated admissibility conditions are placed in order to obtain the uniqueness and stability of such solutions.

The goal of such techniques is to modify the equations by smoothing out the nonlinear terms through a mollification or convolution process. Consequently, higher regularity to the solutions is achieved while, at the same time, capturing the physical behavior of the original model. Mathematically, however, such qualitative properties must be verified in order for this technique to be accepted. More specifically, when applied to a system of partial differential equations usually supplemented with appropriate initial data, the global well-posedness of smooth solutions should be shown. Moreover, the limit of the solutions for the filtered system should converge in some sense to a physical or entropy solution of the original system. In the subsequent sections, we provide the class of functions that will be used as the spatial filters in our method along with the analytical results we have obtained when applying filtering to both the transport model and for the more general quasilinear, symmetric hyperbolic systems of PDEs.

2 Averaging kernels and Filters

Let us describe the appropriate filters we shall implement in the averaging. For a given real-valued function f, we introduce an averaging kernel G and define the filtering of f by the convolution operator

$$\overline{f}(x) = G * f(x) = \int_{\Omega} G(x-y) f(y)\, dy$$

on the given domain $\Omega \subseteq \mathbb{R}^N$.
Given some integer $k \geq 1$, we only consider averaging kernels $G \in W^{k-1,1}(\mathbb{R}^N)$ that satisfy the following:

Properties	Mathematical Expression
Normalized	$\int_{\mathbb{R}^N} G\, d\mathbf{x} = 1$
Non-negative	$G(\mathbf{x}) > 0$ for all $\mathbf{x} \in \Omega$
Symmetric	$\lvert\mathbf{x}_1\rvert = \lvert\mathbf{x}_2\rvert \Rightarrow \lvert G(\mathbf{x}_1)\rvert = \lvert G(\mathbf{x}_2)\rvert$
Non-increasing	$\lvert\mathbf{x}_1\rvert \leq \lvert\mathbf{x}_2\rvert \Rightarrow \lvert G(\mathbf{x}_1)\rvert \geq \lvert G(\mathbf{x}_2)\rvert$

Table 1: The properties of the averaging kernels

In the physical sense, the averaging should only provide non-negative weight to particles, have no preferential direction, and should give more weight to particles that are physically closer. In addition, we prescribe a parameter, $\alpha > 0$, to such a filter such that

$$G^\alpha = \frac{1}{\alpha} G\left(\frac{\mathbf{x}}{\alpha}\right).$$

This parameter α acts as a scaling of the kernel and controls the level of filtering. One example of a commonly studied filter is the Helmholtz filter $f = \overline{f} - \alpha^2 \overline{f}_{xx}$ corresponding to the averaging kernel

$$G^\alpha(x) = \frac{1}{2\alpha} \exp\left(-\frac{|x|}{\alpha}\right).$$

3 Transport equations

Consider the multi-dimensional transport (or transportation) equations which adjoins the continuity equation with the inviscid Burgers equation

$$\begin{cases} \rho_t + \nabla \cdot (\rho \mathbf{u}) = 0 \\ \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = 0 \\ (\rho(\mathbf{x}, 0), \mathbf{u}(x, 0)) = (\rho_0(x), \mathbf{u}_0(x)). \end{cases} \tag{1}$$

The transport equations provide a simple system of two conservation laws for which classical weak solutions may not exist for general initial data, thus a distribution solution consisting of Dirac delta functions (delta-shock solutions) must be introduced [1,2]. On the other hand, the transport equations model the dynamics of particles that adhere to one another upon collision and has been studied as a simple cosmological model for describing the nonlinear formation of large-scale structures in the Universe [3]. For example, $\mathbf{u}$ may represent the flow field carrying dust particles with density ρ, and the delta-shock wave represents a concentration of dust on a shock which attract the dust [4]. The filtering method employed is the observable divergence method and the initial value problem (1) is modified as follows

$$\begin{cases} \rho_t + \overline{\rho} \nabla \cdot \mathbf{u} + \overline{\mathbf{u}} \cdot \nabla \rho = 0 \\ \mathbf{u}_t + \overline{\mathbf{u}} \cdot \nabla \mathbf{u} = \\ (\rho(\mathbf{x}, 0), \mathbf{u}(x, 0)) = (\rho_0(x), \mathbf{u}_0(x)). \end{cases} \tag{2}$$

The following existence result is shown.

Theorem 3.1. *Let $\rho_0 : \Omega \to \mathbb{R}$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^N$ be bounded C^1 functions, then there exist unique classical solutions $\boldsymbol{u}(x,t)$ and $\rho(x,t)$ to the IVP (2).*

4 Quasilinear symmetric hyperbolic systems

In this work we proposed a regularization method for quasilinear symmetric hyperbolic systems through a filtering of the coefficients and the source terms–similar to the filtering described above. The primary motivation here is to generalize and unify the previous results on the Burgers' and homentropic Euler equations while encouraging the application of similar filtering techniques to other physical models other than those found in the study of compressible gas dynamics.

Let us consider a symmetric hyperbolic system of N equations in n-space variables

$$u_t + \sum_i^n A_i(x,t,u)u_{x_i} = h(x,t,u) \quad \text{in } U_T = \mathbb{R}^n \times (0,T), \tag{3}$$

where the A_i's are symmetric $N \times N$ matrices while h, u are N-vector-valued functions. We always prescribe an initial condition to this system

$$u(x,0) = u_0(x). \tag{4}$$

Throughout, we also assume that the system is strictly hyperbolic, i.e. each $N \times N$ matrix $A_i(x,t,u)$ has N distinct eigenvalues

$$\lambda_1^{(i)} < \lambda_2^{(i)} < \ldots < \lambda_N^{(i)}.$$

We introduce spatial averaging to the coefficient matrices in (3) to prevent the finite-time blowup of solutions, thus providing the global well-posedness of smooth solutions to our modified IVP (provided we have sufficiently smooth initial data). More precisely, given a filter, our spatial averaging, regularization technique modifies (3) into the following:

$$u_t + \sum_i^n \overline{A}_i(x,t,u)u_{x_i} = \overline{h}(x,t,u) \quad \text{in } U_T = \mathbb{R}^n \times (0,T), \tag{5}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}^n \tag{6}$$

where the bar represents the convolution product taken with respect to the filter G in the x-variables. More precisely,

$$\begin{aligned} \overline{A_i} &= \left(\overline{a}_{jl}^{(i)}\right) = \left(G^\alpha * a_{jl}^{(i)}(x,t,u)\right) \\ &= \left(\int G^\alpha(x-y)a_{jl}^{(i)}(y,t,u(y,t))\,dy\right) \end{aligned}$$

The averaging for the vector-valued function h is defined similarly. Moreover we impose the condition:

$$\max_{0\le t\le T}\left(\|A(x,t,u(x,t))\|_{1,\infty},\|h(x,t,u)\|_{1,\infty}\right)\le const \tag{7}$$

for any solution $u\in C([0,T],H^k\cap C^1((0,T),H^{k-1})$ for (5)-(6). where $\overline{u}\doteq G*u$ is the convolution product taken with respect to the x-variables. First, we briefly compile several standard results on linear, symmetric hyperbolic systems required in proving the main existence theorems for the quasilinear case. The reader is referred to [5–7] for further details on their proofs.

Linear symmetric hyperbolic systems

The linear system on Euclidean spaces has the form

$$v_t+\sum_{i=1}^n A_i(x,t)v_{x_i}=h(x,t)\ \text{ in } U_T \tag{8}$$

with initial value $v(x,0)=v(0)=v_0(x)$. The following conditions, we refer to by (C1), will be imposed on the linear system.

- (a) $k>1+n/2$ and $v_0\in H^k$.
- (b) A_i are symmetric .
- (c) $t\mapsto A_i(t)\doteq A_i(\cdot,t)$ is of class $C([0,T],H^k(\mathbb{R}^n,\mathbb{R}^{N^2}))$.
- (d) $t\mapsto h(t)\doteq h(\cdot,t)$ is of class $C([0,T],H^k(\mathbb{R}^n,\mathbb{R}^N))$.

Proposition 4.1. *Suppose that*

$$v\in C([0,T],H^1)\cap C^1((0,T),L^2)$$

satisfies the initial value problem to (8), then v satisfies the energy estimate

$$\max_{0\le t\le T}\|v(t)\|_0^2\le e^{CT}\left(\|v(0)\|_k^2+2\int_0^T\|h(s)\|_0^2\,ds\right) \tag{9}$$

where the constant C depends on the supremum of A and its first-order spatial derivatives on U_T.

Another more general estimate is the following.

Proposition 4.2. *Suppose that*

$$v\in C([0,T],H^k)\cap C^1((0,T),H^{k-1})$$

satisfies the initial value problem to (8), then u satisfies the energy estimate

$$\max_{0\le t\le T}(\|v(t)\|_k+\|v_t(t)\|_{k-1})\le C_k e^{\beta_k T}\left(\|v(0)\|_k+\int_0^T\|h(s)\|_k\,ds\right) \tag{10}$$

where the constants C_k and β_k depend on the supremum of A and its spatial derivatives up to order k on U_T.

Proposition 4.3. *The initial value problem to the linear system (8) has a unique solution of class* $C([0,T],H^k)\cap C^1((0,T),H^{k-1})$.

Quasilinear symmetric hyperbolic systems

For the quasilinear case the following conditions will be made and will be referred to by (C2). Define $B_R \subset H^k$ to be the closed ball with radius R.

- (a) $k > 1 + n/2$ and $u_0 \in H^k$.
- (b) For $u \in H^k$, $A(x,t,u)$ and $h(x,t,u)$ are H^k-functions that satisfy (C1).
- (c) The maps $u \in B_R \mapsto A_i(x,t,u)$ and $u \in B_R \mapsto h(x,t,u)$ are bounded (maps bounded sets to bounded sets) and are C^1 maps with bounded derivatives.

Theorem 4.4. *Suppose we have the a priori bound: Given any solution* $u \in C^1(\mathbb{R}^n \times [0,T])$ *of the IVP (3)-(4), there exists a constant* $C_T > 0$ *depending only on* $T > 0$ *such that*

$$\sup_{U_T,|\alpha|\le k} \{|D^\alpha A(x,t,u(x,t))|,\ |D^\alpha h(x,t,u)|\} \le C_T. \tag{11}$$

Then the IVP has a unique classical solution in $C^1(\mathbb{R}^n \times [0,T])$.

The proof is given in three main steps. In step 1, we set up an approximate iteration of linear systems along with a corresponding transformation related to the global solutions to these linear systems. In step 2 we show that this transformation is a strict contraction on an appropriate function space for sufficiently small time. Further, the unique fixed point of this contractive map agrees with the unique short-time classical solution to the quasilinear IVP. Step 3 applies energy estimates along with (11) to extend this unique classical solution further in time.

Proof. (Theorem 4.4) **Step 1:** Choose an arbitrary $T > 0$. We shall prove existence of solutions up to this arbitrary time. First construct the linear problem:

$$v_t + A_i(x,t,u(x,t))v_{x_i} = h(x,t,u) \ \text{ in } U_T \tag{12}$$
$$v(x,0) = u_0(x). \tag{13}$$

where the subscript i is short-hand for summation from 1 to n. The global existence and uniqueness of solutions $v \in C([0,T],H^k)\cap C((0,T),H^{k-1})$ to this IVP holds. The first step to showing existence of a solution to the quasilinear system is to consider the transformation $\mathcal{T}$ defined by $v = \mathcal{T}u$ where u is given and v is the solution to (12)-(13). Our goal is to prove this transformation is a strict contraction on a suitable function space. We consider

$$u \in X^{k,\tau} \doteq C([0,\tau],H^k(\mathbb{R}^n,\mathbb{R}^N)).$$

Using the energy estimates, one has

$$\max_{0\le t\le\tau}\left(\|v(t)\|_k+\|v_t(t)\|_{k-1}\right)\le K_1e^{K_2\tau}\left(\|v(0)\|_k+\int_0^\tau\|h(s)\|_k\,ds\right), \tag{14}$$

where the constants K_i depend only on the constant C_T from (11). Define

$$B_R^{k,\tau}\doteq\{u\in X^{k,\tau}:\ \|u\|_{X^{k,\tau}}\le R\}.$$

It is clear from (14) that $\mathcal{T}$ maps $B_R^{k,\tau}$ to itself for sufficiently small τ and a suitable R. We now show that $\mathcal{T}$ is a contraction on $B_R^{k,\tau}$ in the $X^{0,\tau}$-norm.

Let $v_j=\mathcal{T}u_j$ for $j=1,2$ and set $w=v_1-v_2$. Then w satisfies the linear system

$$w_t+A_i(u_1)w_x=H(x,t)\text{ and }w(0)=0$$

where $H(x,t)=h(x,t,u_1)-h(x,t,u_2)+(A_i(x,t,u_2)-A_i(x,t,u_1))(v_2)_x$. From the Lipschitz continuity of h with respect to u and Sobolev embedding, $\|H(t)\|_0\le C\|u_1-u_2\|_0$ where the constant C depends on R and the Lipschitz constants of A_i and h. Using the energy estimate (9), we obtain

$$\max_{0\le t\le\tau}\|\mathcal{T}u_1-\mathcal{T}u_2\|_0^2\le Ce^{K_2\tau}\tau\max_{0\le t\le\tau}\|u_1-u_2\|_0^2.$$

Hence $\mathcal{T}:B_R^{k,\tau}\mapsto B_R^{k,\tau}$ is a strict contraction with respect to the $X^{0,\tau}$-norm for sufficiently small τ.

Consider the iteration scheme: let $u^{(j+1)}=\mathcal{T}u^{(j)}$ with $u^{(0)}=u_0$. As a consequence of the contraction mapping principle, $u^{(j)}$ converges to a unique $u\in X^{0,\tau}$ i.e.

$$\lim_{j\to\infty}\max_{0\le t\le\tau}\|u^{(j)}-u\|_k=0. \tag{15}$$

Step 2: We show in this step that this limiting function u belongs in $C^1(\mathbb{R}^n\times[0,\tau],\mathbb{R}^N)$. Energy estimates and interpolation inequalities imply that, for any s with $0\le s<k$,

$$\begin{aligned}\max_{0\le t\le\tau}\|u^{(j)}-u^{(l)}\|_s&\le C_k\max_{0\le t\le\tau}\|u^{(j)}-u^{(l)}\|_0^{1-s/k}\max_{0\le t\le\tau}\|u^{(j)}-u^{(l)}\|_k^{s/k} \qquad (16)\\ &\le C\max_{0\le t\le\tau}\|u^{(j)}-u^{(l)}\|_0^{1-s/k}. \qquad (17)\end{aligned}$$

It follows from this and (15) that

$$\lim_{j\to\infty}\max_{0\le t\le\tau}\|u^{(j)}-u\|_s=0$$

for any $0\le s<k$. Choosing s so that $s>1+n/2$, Sobolev embedding implies

$$u^{(j)}\to u\in C([0,\tau],C^1(\mathbb{R}^n)).$$

In addition, it follows immediately from

$$(u^{(j')} - u^{(j'-1)})_t = h(x,t,u^{(j')}) - h(x,t,u^{(j'-1)}) \\ + \; A(x,t,u^{(j'-2)})u_x^{(j'-1)} - A(x,t,u^{(j'-1)})u_x^{(j')}$$

that $u^{(j)}$ converges to u in $C^1([0,\tau], C(\mathbb{R}^n))$, thus $u \in C^1(\mathbb{R}^n \times [0,\tau], \mathbb{R}^N)$.

Step 3: In this step, we extend the local classical solution to the whole interval $[0,T]$. The condition (11) and the previous a priori energy estimate imply that we have a uniform bound $\max_{0\le t\le T} \|u(t)\|_k \le K_T$ for some constant $K_T > 0$. Therefore, we can take the radius of the ball R to be sufficiently greater than K_T then repeat the above local existence argument on $(\tau, 2\tau)$, $(2\tau, 3\tau)$, $(3\tau, 4\tau), \ldots$ until we have covered $[0,T]$. □

Application to spatially averaged systems

Let us apply theorem 4.4 to spatially averaged systems. To do so, we verify condition (11) holds. Given $|\alpha| \le k$, Young's inequality and (7) imply

$$\|D^\alpha \overline{a}_{jl}^{(i)}\|_\infty \le \|G\|_{k-1,1} \|a_{jl}^{(i)}\|_{1,\infty} \le \|G\|_{k-1,1} const.$$

A similar argument holds for h, thus verifying (11). Thus we have shown the following.

Corollary 4.5. *Consider the IVP to the regularized system (5)-(6) along with the same assumptions made on A_i, h, and u_0 as was stated in theorem 4.4 and condition (7). Then there exists a unique global-in-time classical solution for this IVP.*

5 Zero α convergence to weak solutions conservation laws

Our inviscid regularization technique has an important application to conservation laws. We want to study the convergence of the solutions u^α as α limits to zero in order to justify such a regularization method in the sense that it captures the 'behavior' of the original equations. For instance, several important questions arise. Does the sequence of averaged solutions converge in some appropriate function space as the filtering parameter tends to zero? If so, does this limit function satisfy the original, non-averaged system in the weak sense? Consider the following 1d system of conservation laws with forcing

$$u_t + f(u)_x = h(x,t,u) \quad \text{in } U_T = \mathbb{R} \times (0,T) \tag{18}$$

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R} \tag{19}$$

where u, f, h are $\mathbb{R}^N$ vector-valued functions on U_T. Let $A \doteq Df$ and place the same conditions on the resulting quasilinear system

$$u_t + [Df](u)u_x = h(x,t,u) \tag{20}$$

as was given in theorem 4.4. In addition to these previous assumptions, we place further conditions as follows.

Further Assumptions: Let $n = 1$, $k = 2$, $\sup_{U_T} |h(x,t,0)| \le C_0$ for some constant C_0, the initial data has bounded total variation, the Jacobian of the flux f is a diagonal N-by-N matrix, $A(u) = diag(\lambda_1(u), \lambda_2(u), ..., \lambda_N(u))$, and let the filter satisfy

$$\|\overline{u}^\alpha_x(\overline{u}^\alpha - u^\alpha)\|_{L^1_{loc}} = \mathcal{O}\left(\frac{1}{\alpha}\right) \tag{21}$$

for all $u \in L^1_{loc}$

We apply our inviscid regularization to this system of conservation laws to obtain

$$u_t + \overline{A}(u^\alpha)u_x = \overline{h}^\alpha(x,t,u^\alpha). \tag{22}$$

However, we consider a slightly different type of averaging

$$u_t + A(\overline{u}^\alpha)u_x = \overline{h}^\alpha(x,t,u^\alpha). \tag{23}$$

Observe that if one has flux function $f = f(u)$ that is a quadratic polynomial in u (e.g. Burgers' and homentropic Euler equations), then A is linear in u and the two averaged systems are equivalent. Fortunately, for general smooth flux, the global existence theory for the 1d system (23) still holds.

Our aim for this section is to verify that the global classical solutions for the Cauchy problem to (23) does in fact converge to a weak solution for the Cauchy problem to (18)-(19). By a weak solution we mean a solution to the Cauchy problem in the following sense.

Definition 5.1. *A function $u : \mathbb{R} \times [0,T] \mapsto \mathbb{R}^N$ is a weak solution of the Cauchy problem (18)-(19) if u is continuous as a function from $[0,T]$ into L^1_{loc}, the initial condition (19) holds and the restriction of u to the open strip U_T is a distributional solution i.e.*

$$\int_0^T \int_{-\infty}^{\infty} u\phi_t + f(u)\phi_x + h(x,t,u)\phi \, dxdt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)\, dx = 0 \tag{24}$$

for every C^∞ function ϕ with compact support contained in the set $\mathbb{R} \times (-\infty, T)$.

The notion of proving the convergence result is summarized in two key steps. In step 1, the required uniform, BV, and L^1 estimates are established on the sequence of averaged solutions $\{\overline{u}^\alpha\}_{\alpha>0}$ that guarantee compactness in $C([0,\infty), L^1_{loc}(\mathbb{R}))$. In step 2 the limit function in the $\alpha \to 0$ limit is shown to satisfy the definition of a weak solution for the Cauchy problem. Without loss of generality, we prove the case for the scalar case $N = 1$. The proof follows similarly for $N > 1$ as a consequence of the diagonal assumption we imposed on the matrix A.

The following highlights the results needed in obtaining our desired convergence to weak solutions result.

Proposition 5.2. *The sequence of solutions $u^\alpha : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ to the Cauchy problem to (23) satisfy the following bounds.*

$$|u^\alpha(x,t)| \leq M_1 \text{ for all } x, t,$$

$$T.V.(u^\alpha(\cdot,t)) \leq M_2 \text{ for all } t$$

where M_1 and M_2 are independent of α.

With these a priori bounds, we can then show the following.

Theorem 5.3. *The corresponding sequence of averaged solutions $\overline{u}^\alpha : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$ from Proposition 5.2 has a subsequence $\{\overline{u}^\gamma\}_\gamma$ that converges strongly to some function u in $C([0,T], L^1_{loc}(\mathbb{R}))$ as γ limits to zero. Moreover, this limit function u is a weak solution to (18)-(19).*

We should remark that depending on the filter in question, further conditions on the flux f may be necessary in order for (21) to hold. To illustrate this, consider the Helmholtz filter. As the subsequent calculations will show, we must take $f = f(u)$ to be a quadratic polynomial in u.

Using the definition of the Helmholtz filter, we may multiply (23) by a test function ϕ then integrate by parts to obtain

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}} & u^\alpha \phi_t + f(\overline{u}^\alpha)\phi_x + \overline{h}^\alpha(x,t,u^\alpha)\phi \, dx \, dt + \int_{\mathbb{R}} u_0(x)\phi(x,0) \, dx \\
&= \int_0^T \int_{\mathbb{R}} f'(\overline{u}^\alpha)(\overline{u}^\alpha - u^\alpha)\phi_x \, dx \, dt - \frac{\alpha^2}{2} \int_0^T \int_{\mathbb{R}} f''(\overline{u}^\alpha)(\overline{u}^\alpha_x)^2 \phi_x \, dx \, dt \\
&\quad - \frac{\alpha^2}{2} \int_0^T \int_{\mathbb{R}} f'''(\overline{u}^\alpha)(\overline{u}^\alpha_x)^3 \phi \, dx \, dt \\
&\doteq E_1 + E_2 + E_3.
\end{aligned}$$

Clearly, E_1 limits to zero as α tends to zero since $\|\overline{u}^\alpha - u^\alpha\|_{L^1_{loc}} = \alpha^2 \|\overline{u}^\alpha_{xx}\|_{L^1_{loc}} = \mathcal{O}(1/\alpha)$ for the Helmholtz filter and $E_3 = 0$ by our restriction on f. It remains to be shown that the term E_2 limits to zero in the α zero limit. This follows from the following estimate

$$\begin{aligned}
\frac{\alpha^2}{2} & \int_0^T \int_{\mathbb{R}} f''(\overline{u}^\alpha)(\overline{u}^\alpha_x)^2 \phi_x \, dx \, dt \\
&\leq \frac{1}{2} \|f''(u^\alpha)\|_{L^\infty} \|\phi_x\|_{L^\infty} \left(\frac{1}{\alpha} \|u^\alpha\|_{L^\infty} \right) \alpha^2 \int_0^T \int_{\mathbb{R}} \overline{u}^\alpha_x \, dx \, dt \\
&\leq \left(\frac{1}{2} \|f''(u^\alpha)\|_{L^\infty} \|\phi_x\|_{L^\infty} \|u^\alpha\|_{L^\infty} M_2 T \right) \alpha.
\end{aligned}$$

Again, compactness of the sequence of solutions will allow us to formally take the limit as α tends to zero and the limiting function is a global weak solution to the original Cauchy problem.

6 Numerical Simulation

Our regularization technique has been quite successful in regularizing Burgers equations and homenetropic Euler equations [8–14]. In the following we present simulation results for our 1D regularized Equation.

1D Regularized Euler Equations. The equations for the Euler equations without the homentropic assumption are

$$\rho_t + \bar{\rho} u_x + \bar{u} \rho_x = 0 \tag{25a}$$

$$(\rho u)_t + \overline{\rho u} u_x + \bar{u} (\rho u)_x + P_x = 0 \tag{25b}$$

$$(\rho e)_t + \overline{\rho e} u_x + \bar{u} (\rho e)_x + \overline{P} u_x + \bar{u} P_x = 0 \tag{25c}$$

$$P = (\gamma - 1) \left(\rho e - \frac{1}{2} \rho u^2 \right). \tag{25d}$$

With analytical techniques these equations have been proven to converge to weak solutions of the original equation provided modest assumptions on the solutions. We have also found analytical traveling wave solutions for both sets of equations which has led to analytical results showing that shock thickness can be controlled by varying the amount of filtering used in the low-pass filter.

Numerically the equations are shown promising behavior. The solution appear to be well regularized and capture much of the behavior of the original equations. Figure 1 shows a double shock tube simulation for the modified homentropic Euler equations plotted against the solution to the original homentropic Euler equations. Both the expansion wave and shock behavior are being captured. Figure 2 shows a shock tube simulation for the Euler equations. The modified equations are showing a regularized solution that is capturing the expansion wave, contact surface, and shock.

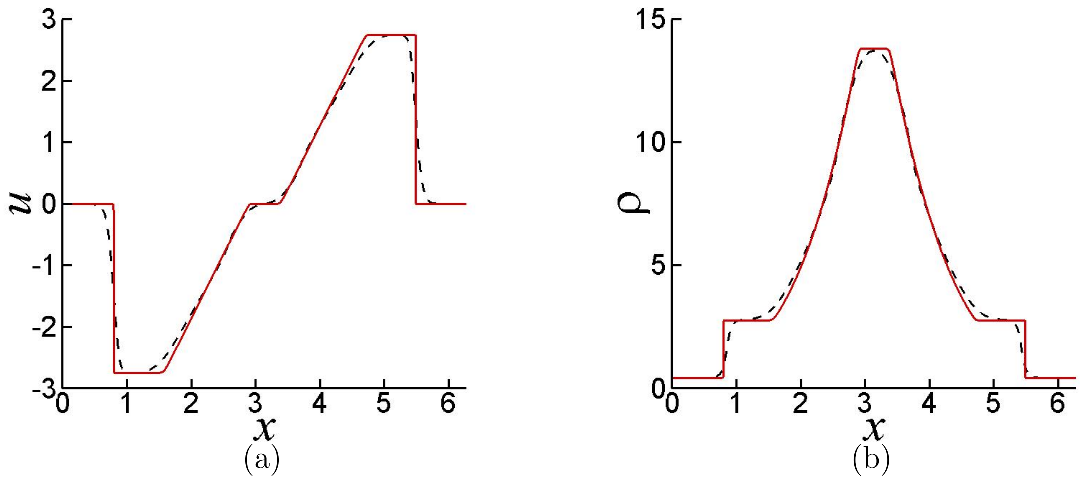

Figure 1: This figure shows a numerical simulation of the modified homentropic Euler equations (dashed line) plotted against the solution to the homentropic Euler equations (solid line). Here the value of $\alpha = 0.05$. In both figures it is clear that the modified homentropic Euler equations are capturing both the expansion wave and shock behavior. (a) The velocity. (b) The density.

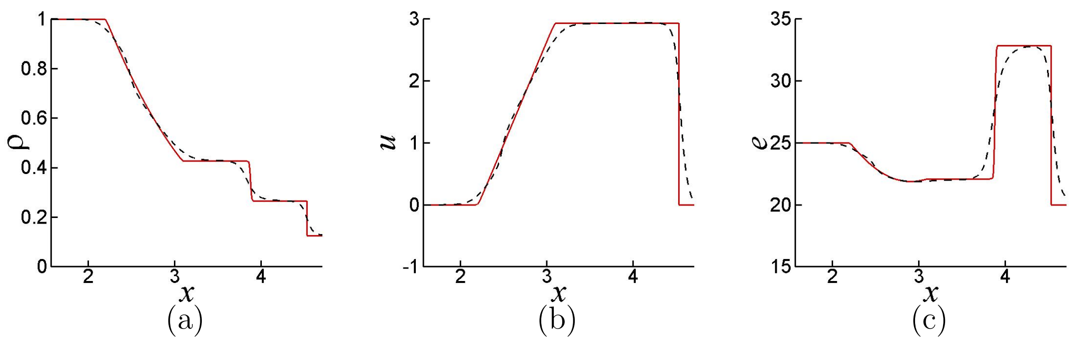

Figure 2: This figure shows a numerical simulation of the modified Euler equations (dashed line) plotted against the solution to the Euler equations (solid line). Here the value of $\alpha = 0.05$. In the figures it is clear that the modified Euler equations are capturing both the expansion wave, contact surface, and shock behavior. (a) The density. (b) The velocity. (c) The energy.

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